Alternative definition of cardinality

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1 Introduction

It seems to me that many mathematicians accept it as an undisputed fact that Georg Cantor conclusively proved that the cardinality of $\mathbb{R}$ is bigger than that of $\mathbb{N}$. Further, we are often told that as a consequence of Cantor's results, infinite sets can be sorted into ascending order based on their *size*.

For example, the set $\mathbb{N}$ is said to have the cardinality $\aleph_0$ and the set $\mathbb{R}$ the cardinality $\mathfrak{c}$. $\aleph_0$ is claimed to be the smallest cardinality in the hierarchy of infinities.

I want make it clear that I consider Cantor's famous diagonal proof as valid in the sense that I think it proves that there exist uncountable sets. In informal language *uncountability* means just that there is no one-to-one correspondence between the elements of an uncountable set and the elements of $\mathbb{N}$.

But ever since high school, I have doubted the existence of infinite cardinalities that supposedly have different sizes. In this article I will suggest an alternative definition of cardinality that I regard as more natural than Cantor's definition. Using the alternative definition, the cardinality of $\mathbb{N}$ and $\mathbb{R}$ is actually proven to be equal.

I suppose I am not completely alone in my doubts, because according to Wikipedia [1]:

The interpretation of Cantor's result will depend upon one's view of mathematics. To constructivists, the argument shows no more than that there is no bijection *between the natural numbers and T. It does not rule out the possibility that the latter are subcountable.*

2 The meaning of "cardinality"

"Cardinality" is basically just a fancy word meaning the same as "size". A finite set such as $A = \{11, 19, 73\}$ has three members, so the size or cardinality of A is three. Let's define set $B = \{20, 40, 60\}$. We see easily that the cardinality of B is also three. These two finite sets are thus considered equal in size.

Let's next think about our everyday life and natural language. Suppose we have two huge baskets filled with peas and that it is our task to find out whether the number of peas is equal in those baskets. One basket is brown, another is black. Imagine that they both contain several thousands of peas.

Would it make sense to start with the following plan:

First, I am going assign each pea in these baskets a unique label. I will label peas in the brown basket with natural numbers starting from one and likewise the peas in the black basket.

Second, using the unique labels, I am going to create a set of ordered pairs, matching each unique pea in the brown basket with a unique pea in the black basket.

Third, if that works out, then I know that the number of peas was equal. Otherwise their number was not equal.

It appears to me that the whole idea of labeling is totally unnecessary if we only want to find out whether we have *an equal number* of peas.

The requirement that the members of the brown basket should be mapped to the members of the black basket, and vice versa, resembles the requirement of having a way to *sort* the elements, because Cantor uses natural numbers as labels and they have a strict ordering.

I think a quite acceptable alternative way of defining cardinality in this case would be the following:

First, pick any one pea from the brown basket.

Second, pick any one pea from the black basket.

Third, throw them away.

Fourth, if both baskets still have at least one pea left, return to the first step.

Fifth, if both baskets are empty, their cardinality was equal. This algorithm halts here.

Sixth, if the brown basket has no peas left, then the cardinality of the black basket was bigger. This algorithm halts here.

Seventh, if the brown basket has at least one pea left, then the cardinality of the brown basket was bigger.

That is the way how we would figure out cardinality in everyday life. Could the same idea be applicable in the realm of set theory where the cardinalities can also be *infinite*? In next section I will argue that the answer is affirmative.

3 Counting cardinality as an active process

To computer scientists and programmers *processes* are a natural way to find out cardinality. By "processes" we mean active methods, i.e. *algorithms*, to perform certain tasks.

For instance, if we have a list data structure and it has no metadata containing the information about the number of elements in that list, then the only way to find out the cardinality could be to *count the members* by using iterators. So we would start with the first element and proceed to the next until there were none left, increasing a counter variable by one along the way.

Let's return to the topic of sets $\mathbb{N}$ and $\mathbb{R}$ again. Peano axioms contain a widely accepted definition that for each member of $\mathbb{N}$, *successor* function $S1$ returns the next member of $\mathbb{N}$. But for the members of $\mathbb{R}$, we cannot define a similar *successor* function $S2$ that would map them to the next member of $\mathbb{R}$.

Now if we wanted to formulate a process that would compare the cardinalities of $\mathbb{N}$ and $\mathbb{R}$ using a method similar to our pea baskets example, how could we proceed? I guess we should have a precisely defined algorithm to remove an element of each set $\mathbb{N}$ and $\mathbb{R}$, for as many steps as it takes to make $\mathbb{N}$ and $\mathbb{R}$ equal to $\emptyset$.

For $\mathbb{N}$, a removeMember algorithm $A1$ is easy to define. First we pick 0, and after that we always pick the successor of the previously removed member.

For $\mathbb{R}$, a removeMember algorithm $A2$ is also easy to define. First we pick 0. We refer to that real number with Z. After having picked Z, we just choose an arbitrary value of, say, 0.1 and add it to the value of the member picked earlier. We keep doing that for all the remaining steps.

In our brown and black baskets example the cardinality of both baskets was finite, so our algorithm also terminated in a finite number of steps. But it is clear that running $A1$ and $A2$, with each step synchronized between them, will never end. $A1$ will keep on going forever and the same is true of $A2$.

Using *this alternative definition* of cardinality, we can conclude that the cardinality of $\mathbb{N}$ and $\mathbb{R}$ is equal, because our mind clearly sees that these infinite “baskets” will never become empty regardless of for how long we keep removing elements from those sets.

To sum it up, the general method of finding out cardinality of set S is:

1. Set integer counter I initial value to 0.
2. Define a removeMember algorithm A for the set S. This algorithm must use a function that removes exactly one member of set S on each successive call.
3. Run algorithm A until the set S equals $\emptyset$. For each step of A, we add 1 to the integer counter I value.
4. If S is finite, then the cardinality of S is the final value of I after algorithm A has halted.
5. Otherwise S must be infinite, but based on our alternative definition of cardinality, we can no longer assign S any kind infinity rank such as $\aleph_0$. We are talking about one single infinity.

4 Responding to criticism

4.1 Non-halting algorithm

Some people could voice an objection and claim that our algorithm A definition is not acceptable, because when S has infinite cardinality, A never halts. Is our alternative definition of cardinality thus invalidated? We do not think so at all.

It is a well-known basic fact of set theory that all *finite* sets can always be defined by *enumeration* even though that method can be painfully impractical in many cases. But remember how *infinite* sets are defined or "constructed". Quite obviously all definitions involving infinite sets are made by describing *algorithms that never end*, too.

For instance, the members of $\mathbb{N}$ just *cannot* be defined by using enumeration - at least in this world that we now inhabit. That is why criticizing *our infinite algorithm* based on the fact that it never halts in the case of infinite sets is *not* fair, unless you are also objecting constructing infinite sets to begin with.

During infinite set construction time, we are expected to "grasp" the meaning of a set of, say, $\mathbb{N}$. But it is clear to us that our alternative definition of cardinality requires no more than using a very similar method to that which we have already accepted. In this proposal we are only expected to see that *removing elements from an infinite set is always a process that never ends* just like when constructing the set of $\mathbb{N}$ we are expected to grasp that the sequence of natural numbers never ends.

4.2 How do we know whether algorithm A will halt or not?

Another possible objection is that in order to find out the cardinality of S, we have to run algorithm A, but our alternative definition of cardinality does not perhaps explain in enough detail *when* A halts or not. How can we know that?

To answer that question, we only have to examine the original *definition* of set S that we are trying to empty in order to count the cardinality of S.

The outcome of algorithm A completely depends on that. It suffices to consider exactly two cases:

1. If the definition of set S can be expressed using enumeration, then S must finite and then we know that our algorithm A will always halt in a finite number of steps when emptying set S using removeMember function.
2. Otherwise, if the definition of set S *cannot* be expressed using enumeration, then it must be a definition of a set S that is not finite. In this case the definition of S must be e.g. something like Peano's *successor* function

that defines a never-ending sequence of natural numbers. So based on the knowledge of the definition of set S, we automatically also know that we cannot empty the set S using a finite number of removeMember invocations. Thus the set S is infinite.

4.3 Mathematics based on the traditional definition of cardinality

Ever since Cantor's results concerning the hierarchies of infinities were published, many mathematical geniuses have been overjoyed with the notion of having been granted access to "Cantor's paradise". Rejecting the traditional definition in favour of the alternative definition proposed in this paper would imply that a huge amount of existing mathematical results would be invalidated or at any rate their correctness would have to be proven again using different methods.

Even if this alternative definition were considered to be simple and intuitive by *some* logicians and mathematicians, there would still be considerably more resistance towards it. Like we said, this is simply because so much important mathematical research assumes that Cantor's definition of cardinality was correct. Abandoning all those dearly treasured mathematical results would likely amount to a severe crisis in mathematics. Cardinality is such a fundamental building block of many other theories.

But when we examine the history of mathematics, we realize that Euclidian geometry was once regarded as totally self-evident and undisputed. For several *centuries*, thinking otherwise would have been considered completely crazy. Nobody even attempted to so. Nevertheless, non-Euclidian geometry later emerged.

In the field of physics, Einstein's ideas made Newton's classical "self-evident" and universally accepted results outdated. Likewise in physics, advances in quantum research have made us consider whether classical logical principles, once thought completely clear and self-evident as well, would have to be re-evaluated.

We have no delusions of grandeur or egotistical reasons for pushing our case forward. The sole purpose of this article is to describe our alternative definition and see how thinking people react. To be clear, we consider the simplicity of this alternative definition as a *strength*, not a weakness. So we are not ashamed to publish our result and let others use their own reasoning to decide the possible value of it.

References

[1] Cantor's diagonal argument. https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argu
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